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NAVIGATION SYSTEMS

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B. E. Trotter

Countermeasures Branch
Radio Division

October 22, 1957

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MOON DOPPLER NAVIGATION
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ABSTRACT
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The science of navigation has been augmented by the use of radio aids, of which a recently considered one has been that of utilizing radio-frequency energy reflected from the moon. Two characteristics of such a signal have been examined to determine their applicability in a terrestrial navigation system; first, the Doppler frequency shift undergone by the signal, and second, the time rate of change of Doppler shift.

The ratio of Doppler frequency shift to the time rate of change of Doppler shift is a function of the longitude of the observer, and the ratio of Doppler shift to the sine of the local hour angle of the moon is proportional to the latitude of the observer.

The probable error in position on the surface of the earth, resulting from the probable errors in the parameters of an idealized system, range from a few meters at latitudes near 90 degrees to approximately 100 kilometers at equatorial latitudes; however, nagivation to within line of sight (10 km) of a given position may be accomplished over 70 percent of the hemisphere illuminated by the moon at a given time. The position error is a function of position as well as time and is relatively independent of the transmitter frequency employed. The ultimate limitations in position error are those due to the fundamental uncertainty in relative velocity between earth site and moon center and to the probable error in transmitter frequency. The ratio of probable error in transmitter frequency to the transmitter frequency should be of the order of 1 part in 10^{11} if the full capabilities of the system are to be realized.

Two experiments were performed at field sites of the Laboratory as part of this investigation and the results indicate that the position errors described here are of the right order of magnitude.

PROBLEM STATUS

This report completes one phase of the problem; work on other phases continues.

AUTHORIZATION

NRL Problem R06-13
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Manuscript submitted August 7, 1957

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MOON DOPPLER NAVIGATION
[Secret Title]

INTRODUCTION

The first successful radar contact with the moon was reported in 1946 by the United States Army (1). The years following this event have been marked by increasing interest in the reflection of radio waves from the moon. It has been proposed that techniques developed in the course of experiments conducted in this branch of radio astronomy be employed to achieve the following ends:

1. determine the earth-moon distance,
2. measure the relative velocity of earth and moon,
3. obtain the velocity of propagation of radio waves in the earth-moon space, and
4. investigate the possible applications of data obtained during experiments of this nature.

This report had its origin in 4. Its purposes are to:

1. study the Doppler shift in the received frequency of a radio-frequency transmission from earth to moon to earth as a function of earth rotation and orbital motion of the moon, and
2. investigate the possibility of establishing a terrestrial navigation system based on contours of constant Doppler frequency.

The navigation system to be analytically studied is one where a single site is used for transmission and reception. This arrangement is termed "monostatic." Following the analytic treatment, limited experimental results are given for a bistatic arrangement where the transmitting and receiving sites are separated by 25 miles.

The term "navigation" implies the calculation of position and direction with minimum error. If there is a given error in the location of a fixed site on the surface of the earth, and the Doppler measuring system is free of error, the uncertainty in the location of the position of the Doppler measuring site relative to the position of the fixed site is equal to the given error. If, however, the Doppler measurement introduces error, the uncertainty in location is amplified. The errors introduced by the Doppler measuring technique, and the magnitude of the resulting amplification of position uncertainty on the surface of the earth, will be examined.

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THEORY

When a radio-frequency signal is transmitted from the earth, reflected by the moon and received on the earth, it undergoes a double Doppler frequency shift, one on going and another on the return journey. The total shift (Appendix B) is given by the expression

$$\Delta f = f_r - f_t \tag{1}$$

$$\Delta f \approx f_t \left(\frac{2\,v}{c}\right) \tag{2}$$

where

f_r is the frequency received on the earth

f_t is the frequency transmitted from the earth

v is the relative velocity of the site on the earth and the moon

c is the velocity of propagation in the earth-moon space

Δf is the Doppler frequency shift.

The Doppler frequency shift is a consequence of the earth's rotation and the moon's orbital motion. It is, as shown in Eq. (2), proportional to the relative velocity between the moon and the receiving site on the earth.

A view of the earth-moon system, from a point above the earth's north geographic pole, is shown in Fig. 1,

where

H is the earth's north geographic pole

A is a receiving site located on the earth's equator

L is the moon

R is the earth's radius

ω is the earth's angular velocity of rotation

LHA is the angle between HA and the line connecting earth and moon centers (this is called the local hour angle of the moon).

If the moon lies in the earth's equatorial plane (declination 0 degrees), the relative velocity of the site A with respect to the moon (due to the rotation of the earth) is $-\omega R \sin \mathrm{LHA}$ (a positive velocity exists when the distance between earth site and moon center is diminishing). Let a plane be constructed so that it includes the earth's polar axis and the line connecting earth and moon centers. All points on the earth lying in a plane drawn through site A, parallel to the plane just described, will have equal relative velocities with respect to the moon (since the earth is a rigid body). However, this is strictly true only if the moon is stationary, relative to the earth's center and the fixed stars, and at an infinite distance. This condition is assumed to avoid introducing complex factors at this point. Figure 2 is a view of the earth from a direction mutually perpendicular to the earth's polar axis and the line connecting earth and moon centers, and

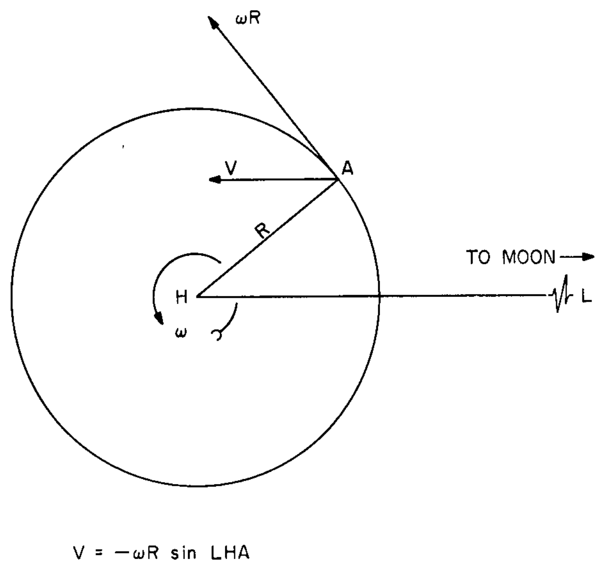

Fig. 1 - Relative velocity of site A with respect to moon

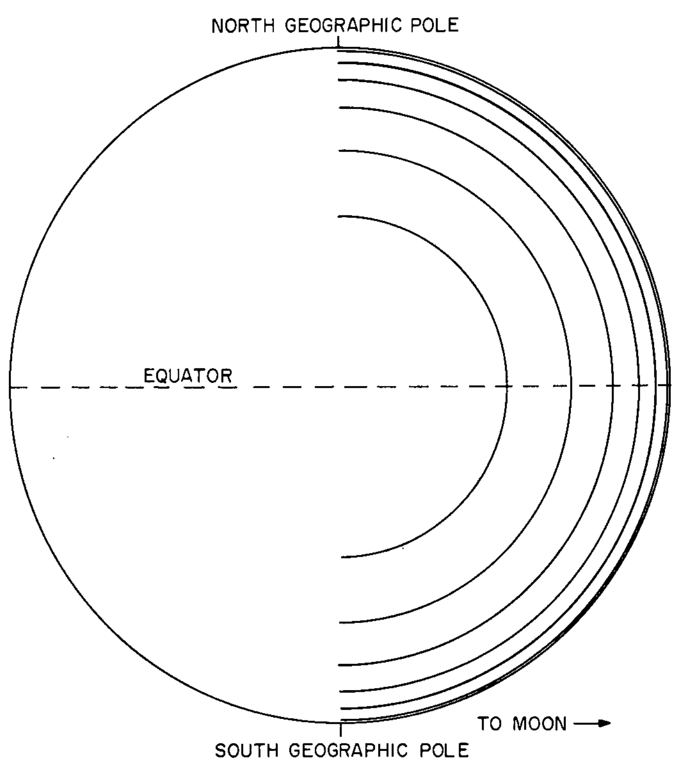

Fig. 2 - Semicircles of equal velocity on the earth's surface with respect to moon, viewed from zenith of geographical point having latitude 0 degree and local hour angle 270 degrees

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illustrates the intersection of planes of equal velocities and the earth's surface. In this view, where the meaningful curves of equal velocity are coaxial semicircles, the point of maximum velocity is the point of intersection of the earth's surface and the line through the centers of the semicircles, that is, the point on the earth's surface having latitude 0 degree and local hour angle 270 degrees. Proceeding around the earth from this point one encounters semicircles of diminishing velocity until the meridian, whose plane includes the moon, is reached where the relative velocity with respect to the moon is zero. Since the Doppler frequency shift is proportional to the relative velocity, these curves are contours of constant Doppler shift. They may be thought of as a fixed grid or cage of constant Doppler frequency lines in which the earth rotates.

To introduce the concept of moon Doppler navigation, Fig. 3 is presented. This is a view of the earth-moon system from a point above the earth's north geographic pole. The parallel lines represent the intersection of equal velocity planes with the surface of the earth. The relative velocity, with respect to the moon, of an observer moving from point A to B to C will vary sinusoidally with respect to the local hour angle. The relative velocity of an observer at an intermediate latitude, moving from D to E to F, will also vary sinusoidally, but at lower amplitude. The amplitude at an intermediate latitude is reduced by the cosine of the latitude L. This leads to an expression of relative velocity of observer with respect to the moon of

$$V = -\omega R \cos L \sin LHA. \tag{3}$$

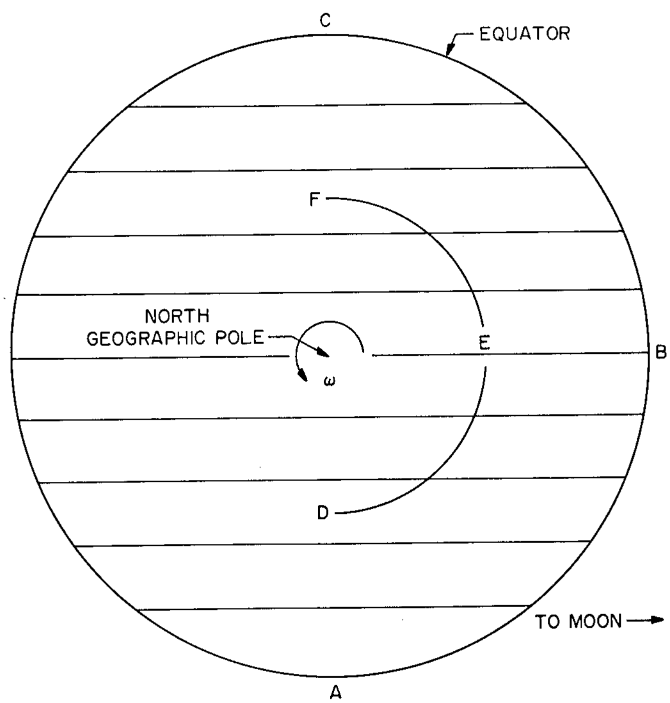

Fig. 3 - Lines of equal velocity with respect to moon, viewed from north celestial pole. Plane of paper is earth's equatorial plane. Parallel lines are intersection of surface of earth and planes normal to paper.

Figure 4 illustrates a family of curves of relative velocity as a function of local hour angle. Figure 5 shows a family of curves of rate of change of relative velocity with respect to local hour angle. Since the Doppler frequency shift is proportional to relative velocity, these curves pertain equally to Doppler shift, and, by substitution of velocity from Eq. (3) into the Doppler shift expression, Eq. (2),

$$\Delta f = -\frac{2 f_t}{c} \omega R \cos L \sin \mathrm{LHA}. \qquad (4)$$

Differentiation with respect to local hour angle yields

$$\frac{d\,\Delta f}{d\,\mathrm{LHA}} = -\frac{2 f_t}{c} \omega R \cos L \cos \mathrm{LHA}. \qquad (5)$$

Since ω is equal to $d\,\mathrm{LHA}/dt$, the time rate of change of Δf may be found by multiplying Eq. (5) by ω. If, at the same time, Eq. (4) is also multiplied by ω, the following pair of parametric equations related by the local hour results:

$$\frac{d\,\Delta f}{dt} = \dot{\Delta f} = -\frac{2 f_t \omega^2 R \cos L \cos \mathrm{LHA}}{c} \qquad (6)$$

$$\omega\,\Delta f = -\frac{2 f_t \omega^2 R \cos L \sin \mathrm{LHA}}{c}. \qquad (7)$$

These are the equations of a sphere of radius, $2 f_t \omega^2 R/c$. Dividing Eq. (7) by Eq. (6), one obtains

$$\frac{\omega\,\Delta f}{\dot{\Delta f}} = \tan \mathrm{LHA}.$$

West longitude is determined by subtracting the local hour angle from the Greenwich hour angle; east longitude is obtained by subtracting the Greenwich hour angle from the local hour angle. In symbolic form,

$$\text{Long W} = \mathrm{GHA} - \mathrm{LHA}, \text{ and}$$

$$\text{Long E} = \mathrm{LHA} - \mathrm{GHA}.$$

It may be necessary to add 360 degrees to either quantity. The Greenwich hour angle of an object is found by subtracting the right ascension (RA) of the object from the Greenwich sidereal time (GST) of the observation. The Greenwich hour angle of the moon may be found in the Air Almanac (or equivalent tables).

When the value of LHA has been determined as above, Eq. (4) may be used to obtain the latitude of the observer.

$$L = \arccos \frac{c\,\Delta f}{2 f_t \omega R \sin \mathrm{LHA}}.$$

A nomograph, furnishing a graphical procedure for the solution of Eqs. (6) and (7) is shown in Fig. 6. This, like Fig. 3, is a view of the earth from the extended polar axis. There is a rectangular reference system having an ordinate of $-\omega\Delta f$ and an abscissa of $-\dot{\Delta f}$. An observer on the earth, capable of determining the Doppler frequency shift Δf,

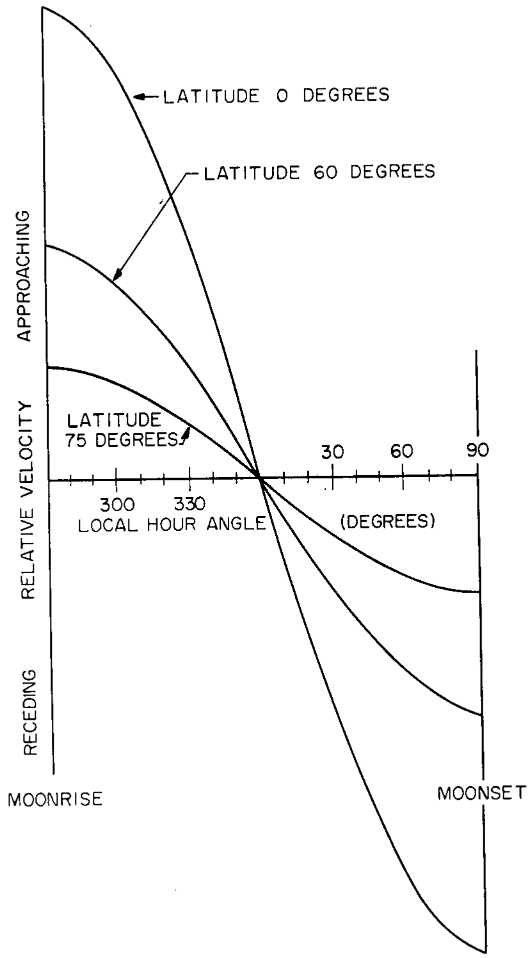

Fig. 4 - Relative velocity with respect to moon as a function of local hour angle

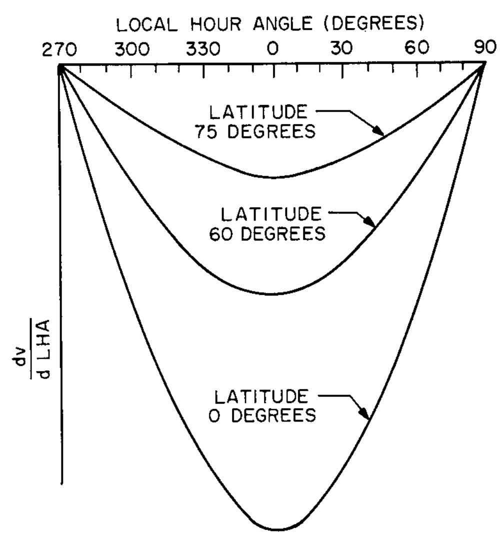

Fig. 5 - Relative acceleration with respect to moon as a function of local hour angle

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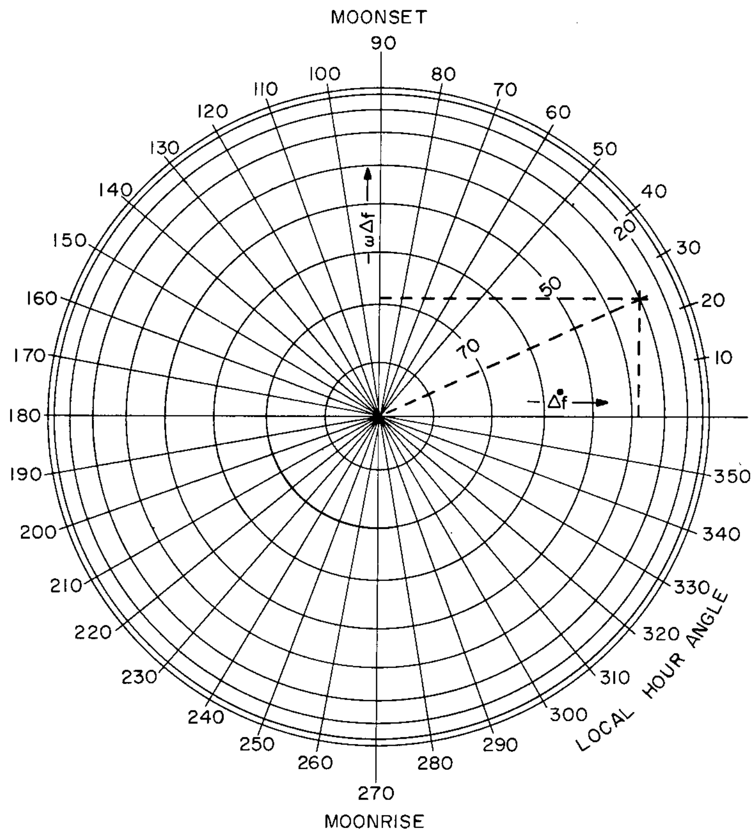

Fig. 6 - Moon Doppler navigation nomograph

the time rate of change of Doppler shift $\Delta\dot{f}$, and the location in the frequency domain of the returned signal with respect to the transmitted signal, would enter the nomograph and determine his latitude and local hour angle with respect to the moon. The nomograph is used in the following manner:

1. Determine the maximum radius of the nomograph as $r = 2\ f_t\ \omega^2\ R/c$.

2. Normalize the product of earth's angular velocity and the measured value of Doppler shift $\omega\Delta f/r$. This equals the product of $\cos L \sin LHA$ and is the ordinate of the earth site on the nomograph.

3. Normalize the measured value of time rate of change of Doppler shift $\Delta\dot{f}/r$. This equals the product of $\cos L \cos LHA$ and is the abscissa of the earth site on the nomograph.

A point having a given ordinate and abscissa is shown in Fig. 6.

This particular site has a latitude of 30 degrees N and a moon local hour angle of 25 degrees. The radial lines from the origin in Fig. 6 represent the local hour angle of the moon from the site. The fourth quadrant, in the mathematical sense, represents local

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hour angles from moonrise (LHA = 270 degrees) to the local meridian (LHA = 0 degree). The first quadrant represents local hour angles from the local meridian to moonset (LHA = 90 degrees). The second and third quadrants represent local hour angles from 90 to 270 degrees and are required when the declination of the moon varies from zero degr

The concentric circles are parallels of latitude for the northern hemisphere as shown in the nomograph. A view of the nomograph from below the plane of the paper would represent the southern hemisphere (in other words, there is a hemispheric ambiguity in the nomograph and in the solution of the equations on which it is based).

A simplified system has been assumed for purposes of clarity. The existing situation is more complex than the one just described. The moon's orbital velocity, changing distance from the earth, and varying declination enter the calculation of relative velocity. In addition, the transmitter may be widely separated from the receiver (bistatic operation). Further discussion of these effects appears in Appendix D. The principles described above are not changed even when the complexity of the existing earth-moon system is considered. A system of measurements based on these principles would give a continuous flow of position data during the time from moonrise to moonset. Such a system could be used as the base of a terrestrial navigation system, independent of weather (which affects celestial navigation by optical means).

PROBABLE ERROR IN POSITION DETERMINATION

When the value of a property is to be computed from observations of other properties, it is natural to inquire about the exactness with which each observation should be performed to produce a given degree of precision in the result. Precision expresses the degree of concordance or consistency within a set of observed values. The best representation of the degree of consistency in a set of discordant measurements of a quantity which were performed with equal exactness is the probable error. The probable error of a quantity may be defined as that error which is just as likely to be exceeded as not to be exceeded. The probable error of a computed result is derived in Appendix C.

The position of a site on the surface of the earth is calculated from measurements of Doppler frequency shift and time rate of change of Doppler shift, which in turn, are dependent on the values of relative velocity and acceleration of the site with respect to the moon. If the probable error in position is desired, it is necessary to determine the probable error in both the Doppler frequency shift and the time rate of change of Doppler shift. This is done in Appendix E.

The equation of relative velocity of a site on the earth with respect to the moon is derived in Appendix D. The derivation of an equation describing the relative acceleration of the site with respect to the moon from the complete velocity equation is a formidable task, however, it is illuminating to develop the probable errors in relative velocity and acceleration from simplified velocity and acceleration equations based on earth rotation alone. The order of magnitude of errors in position obtained in this case may be said to represent the minimum probable errors to be expected. This analysis should indicate the degree of usefulness of the Doppler method of navigation.

The probable errors in relative velocity and acceleration due to the rotation of the earth are derived in Appendix E, and the position errors in latitude and longitude are derived in Appendix F. The errors in position are functions of position as well as time. The extreme values of position errors in latitude, as a function of latitude, for local hour

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angles of 0 and 90 degrees are shown in Fig. 7. The longitude errors, for local hour angles of 0 and 90 degrees as a function of latitude are shown in Fig. 8. The "assumed geodetic error in position" is based on the assumption that a position on the surface of the earth may be determined to within plus or minus ten feet in three mutually perpendicular directions by means of geodetic and gravimetric surveys.

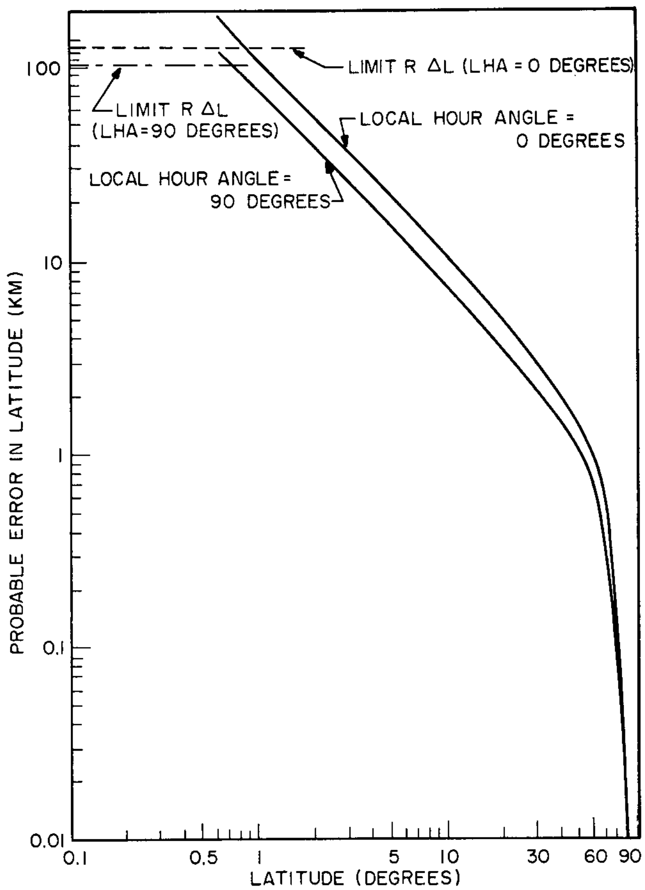

Fig. 7 - Latitude error as a function of latitude

A plot of contours of constant probable error in latitude (in kilometers), when the geographical point of the moon is at latitude 0 degree and longitude 0 degree, is shown in Fig. 9. This plot shows that navigation to within line of sight of a given position may be accomplished over 70 percent of the hemisphere visible to the moon. This a clear indication of the degree of usefulness of moon Doppler navigation.

TRANSMITTER FREQUENCY STABILITY

If the Doppler frequency shift is determined by beating the transmitter frequency with the echo, the quantity actually measured at the earth site is

$$\Delta f' = f_t \left(1 + \frac{2v}{c}\right) - f_t'$$

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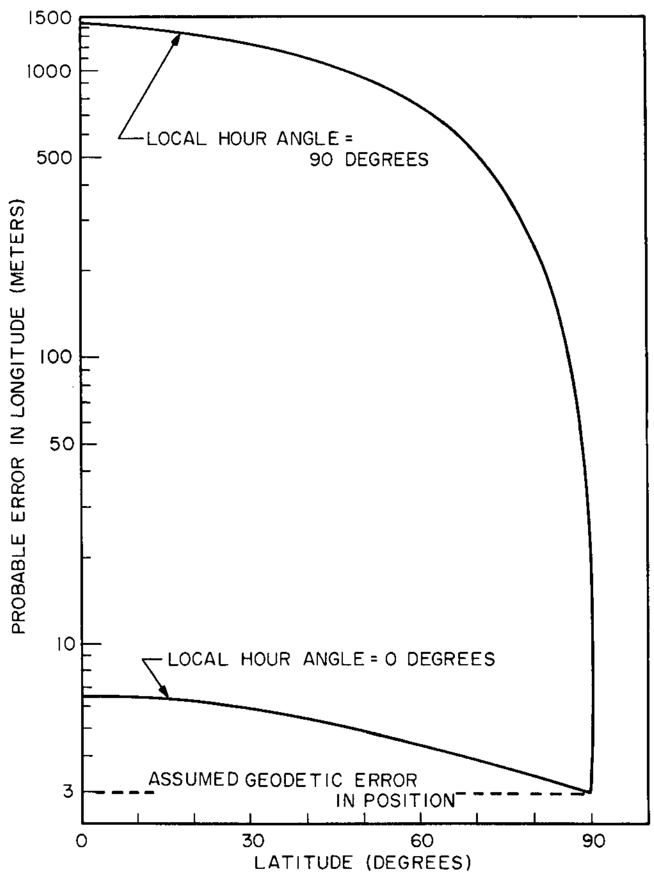

Fig. 8 - Longitude error as a function of latitude (declination = 0 degrees)

where

f_t is the transmitter frequency at the time of transmission

f_t' is the transmitter frequency at the time of reception of the echo

v is the relative velocity of site and moon, and

c is the velocity of propagation.

The measured value of Doppler shift is a function of four variables, $\Delta f' = G(f_t, v, c, f_t')$, and the probable error in the measured value of Doppler shift is given by

$$\epsilon_{\Delta f'} = \sqrt{\left(\frac{\partial \Delta f'}{\partial f_t} \epsilon_{ft}\right)^2 + \left(\frac{\partial \Delta f'}{\partial v} \epsilon_v\right)^2 + \left(\frac{\partial \Delta f'}{\partial c} \epsilon_c\right)^2 + \left(\frac{\partial \Delta f'}{\partial f_t'} \epsilon_{ft'}\right)^2}$$

or

$$\epsilon_{\Delta f'} = \sqrt{\left(1 + \frac{2v}{c}\right)^2 \epsilon_{ft}^2 + \left(\frac{2 f_t}{c}\right)^2 \epsilon_v^2 + \left(-\frac{2 f_t v}{c^2}\right)^2 \epsilon_c^2 + \epsilon_{ft'}^2} .$$

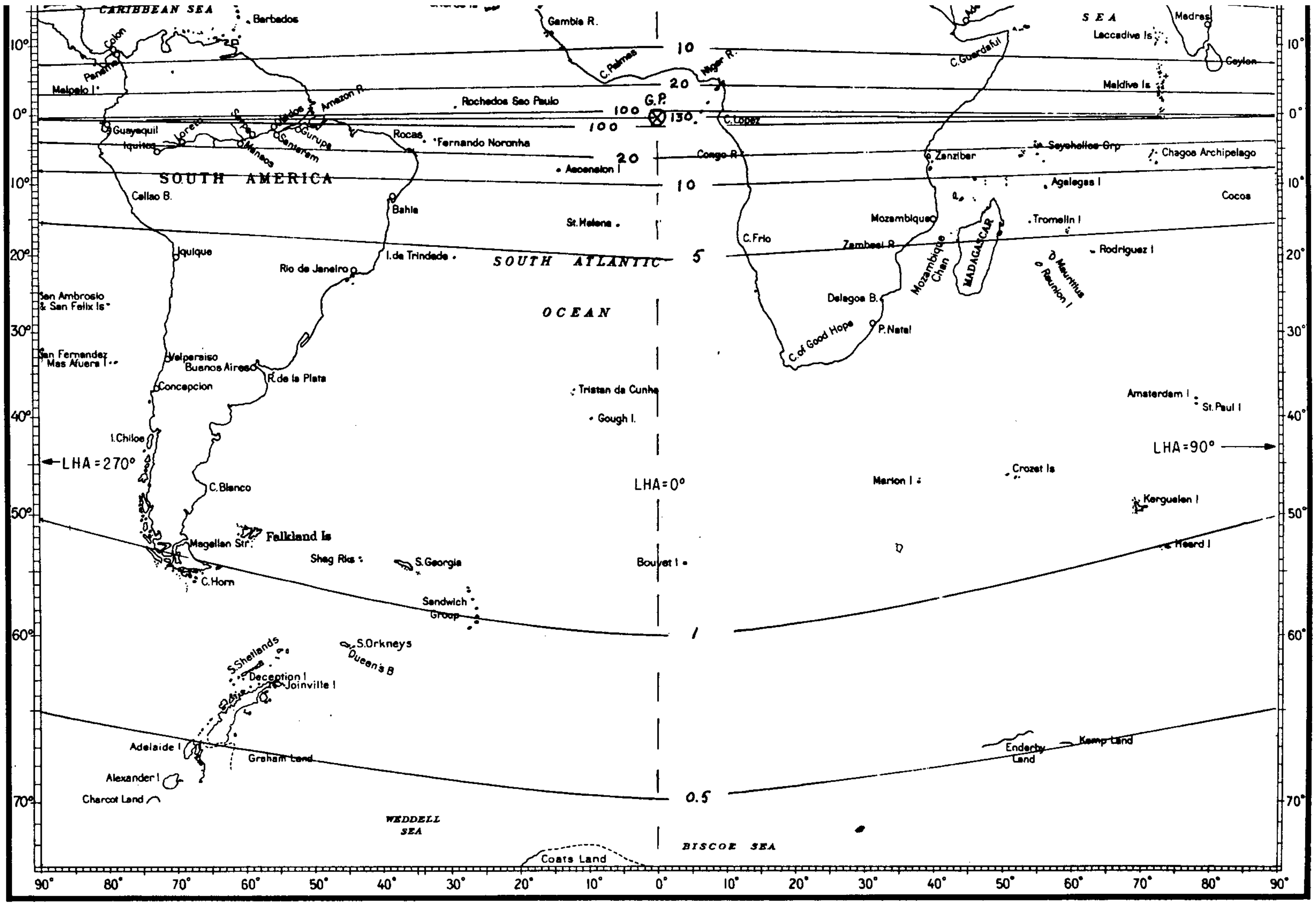

Fig. 9 - Contours of constant probable error in latitude (in kilometers) when the geographical point of the moon is at latitude 0 degrees and longitude 0 degrees

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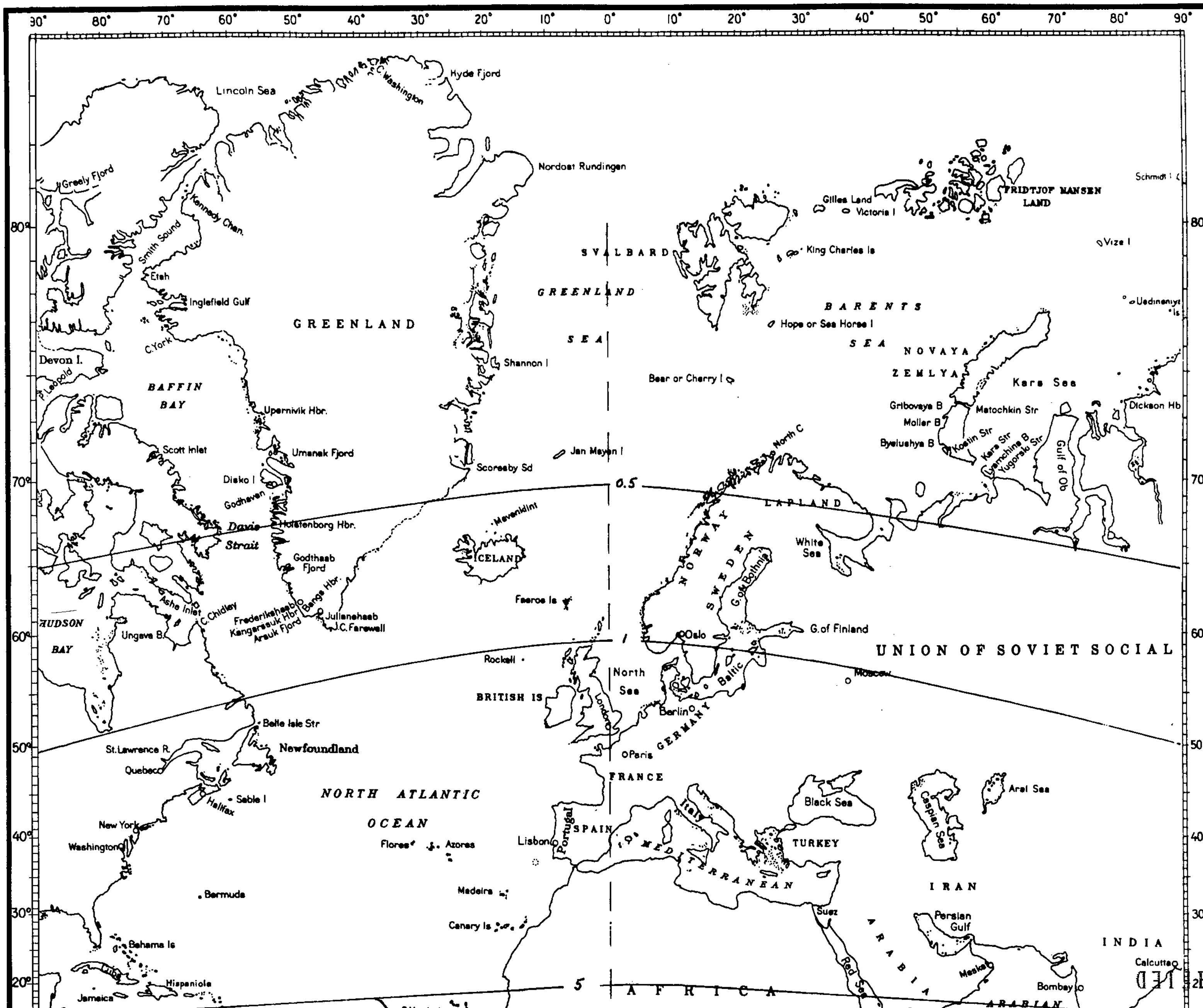

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Since $v/c << 1$, this expression reduces to

$$\epsilon_{\Delta f'} = f_t \sqrt{2\left(\frac{\epsilon_{ft}}{f_t}\right)^2 + \left(\frac{2\,\epsilon_v}{c}\right)^2}$$

An examination of the range of values taken on by the probable error in relative velocity (Appendix E) gives these values as extremes at low and high latitudes respectively:

$$\epsilon_v(\max) \doteq 6.4 \times 10^{-2} \text{ meters/sec}$$

$$\epsilon_v(\min) \doteq 2.2 \times 10^{-4} \text{ meters/sec.}$$

It can be seen from the expression for the probable error in measured Doppler shift $\epsilon_{\Delta f'}$, that the ratio of probable error in transmitter frequency to transmitter frequency should be an order less than the ratio of probable error in relative velocity to the velocity of propagation, if the probable error in measured Doppler shift is to be primarily dependent on the latter ratio. If this is the case, the requirement of transmitter frequency stability in this system may be expressed as

$$\frac{1}{10^{13}} \leq \frac{\epsilon_{ft}}{f_t} \leq \frac{1}{10^{11}},$$

however, at the higher latitudes (where the frequency stability requirements approach 1 part in 10^{13}) the error in position diminishes, and for a specified error in position the stability requirements may be relaxed.

LIBRATION EFFECT

If the surface of the moon is rough and scatters radiation in all directions, there will be an increase in the directivity of the scattered radiation and of the echo cross section compared to that from a smooth moon. Energy may then be returned from the whole visible disk and not only from the first Fresnel zone. This introduces, if the moon is rotating, a Doppler spread (equal to $4\ v/\lambda$) in the echo spectrum corresponding to plus or minus the velocity of approach v of the limb relative to the center (2). The angular velocity of libration of the moon reaches a maximum of about 3 degrees per day, hence, the velocity of approach of the limb is of the order of one meter per second. Although the position errors in latitude and longitude are shown (Appendix F) to be independent of transmitter frequency when the Doppler shift due to earth rotation alone is employed, the libration effect, as a source of error, would increase with increasing frequency. However, good evidence has been obtained recently which indicates a reflecting zone much smaller than the visible disk of the moon. This would greatly reduce the width of the echo spectrum.

EXPERIMENT

Two experiments were performed, one on May 13, 1956, the other on July 7, 1956, at field sites of the laboratory for the purpose of measuring the Doppler frequency shift undergone by a 301-Mc transmission from earth to moon to earth. Each experiment had a duration of approximately thirty minutes because of the limited aspect of the transmitting

antenna. The transmitting site comprised a 10-kw, cw, 301-Mc transmitter and a fixed parabolic antenna having a gain of 37 db at this frequency. The physical aperture of the antenna had dimensions of 220 ft by 263 ft and is shown in Fig. 10. Limited steering was accomplished by moving the antenna-feed system on either side of the focus of the paraboloid within a region of low aberration. The receiving site included a receiver having a noise figure of 6 db fed by a 25-ft paraboloid with a gain of 25 db. The receiving antenna is shown in Fig. 11. The system parameters were:

Transmitter power 10 kw	+40 dbw	
Transmitting antenna gain	+37 db	
Receiving antenna gain	+25 db	
Path attenuation	-258 db	
Available signal power at receiver	-156 dbw	-156 dbw
Noise power/cycle (kT) 4.14×10^{-21} watts	-204 dbw	
1-kc bandwidth 10^3 cps	+30 db	
Noise figure of receiver	+6 db	
Noise power/kc bandwidth (signal power for S/N ratio of 0 db)	-168 dbw	-168 dbw
Estimated signal to noise ratio -156 - (-168)		+12 db

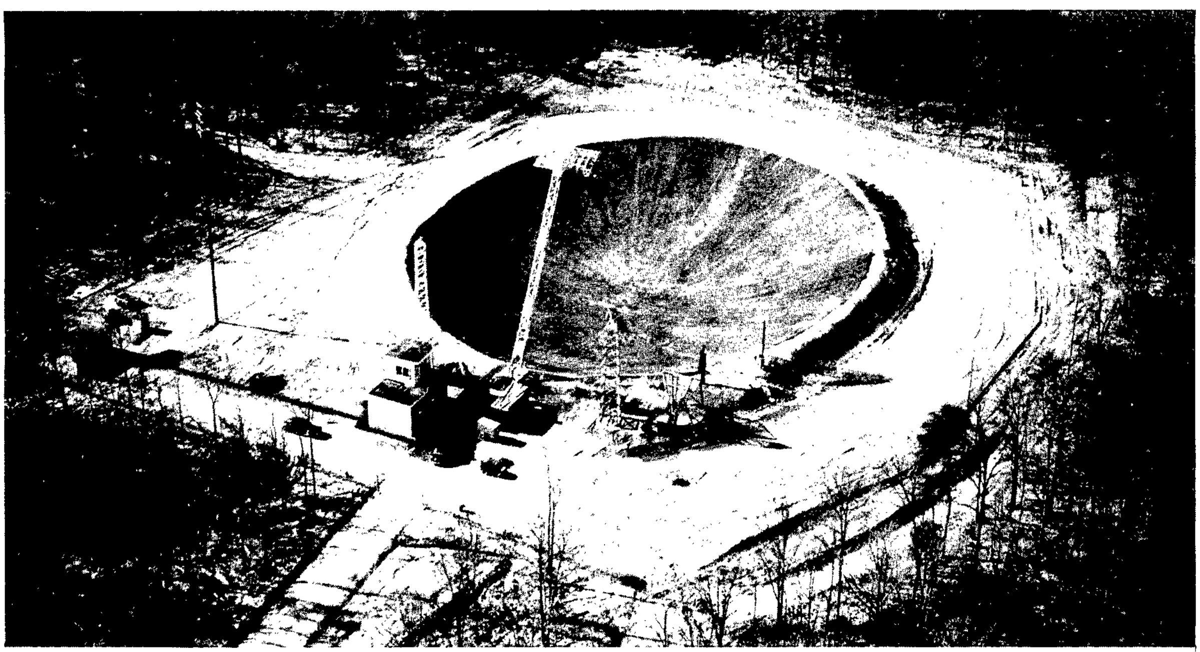

Fig. 10 - Transmitting antenna

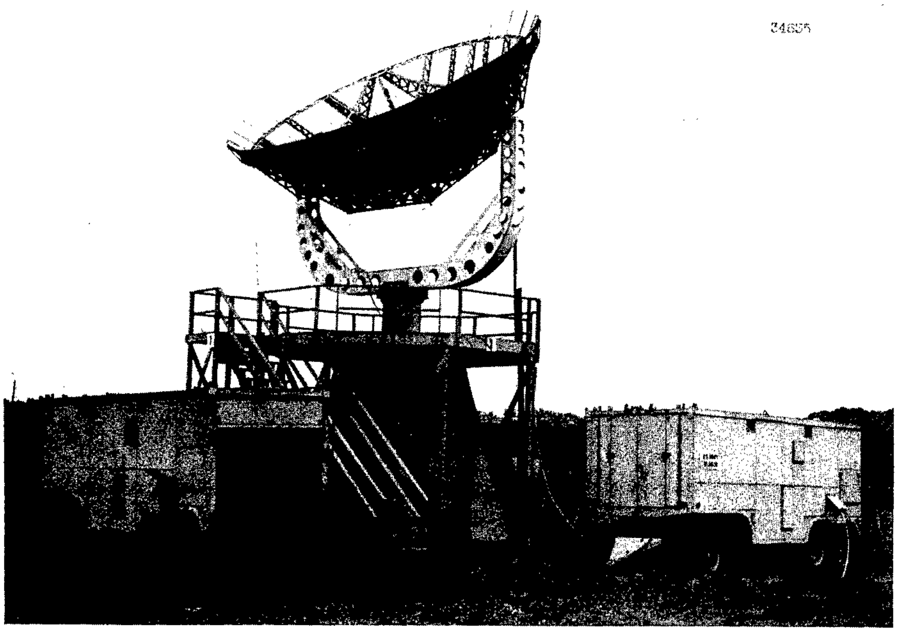

Fig. 11 - Receiving antenna

The Doppler frequency shift was obtained by beating the returning signal with the ground wave arriving at the receiving site from the transmitting site 25 miles distant, Fig. 12. An audio oscillator was tuned to the beat frequency by visual observation of Lissajous figures on an oscilloscope. An electronic counter was employed to measure the audio oscillator frequency. The measured values of the Doppler shift, as a function of time, are compared with the computed-value curves in Figs. 13 and 14.

The absolute value of the average error (or deviation of measured values from the computed curve) is approximately 4 cps. The computed value of probable error in Doppler shift (under the experimental conditions) is 0.1 cps. The application of "the method of least squares" to find the best linear representation of the experimental values, results in a curve which agrees with the computed curve within plus or minus 0.1 cps.

CONCLUSIONS

The location of a fixed site on the earth may be determined within plus or minus 10 feet in three mutually perpendicular directions by means of geodetic and gravimetric surveys. These surveys are performed with great care and require considerable data. It is logical to expect the precision of a navigational system to be proportional to the quantity and quality of data employed. It has been the purpose of this report to examine and evaluate the quality of available data employed in a moon Doppler navigation system.

The fundamental quantities employed to determine position on the surface of the earth, in this system, are the Doppler frequency shift and the time rate of change of Doppler shift

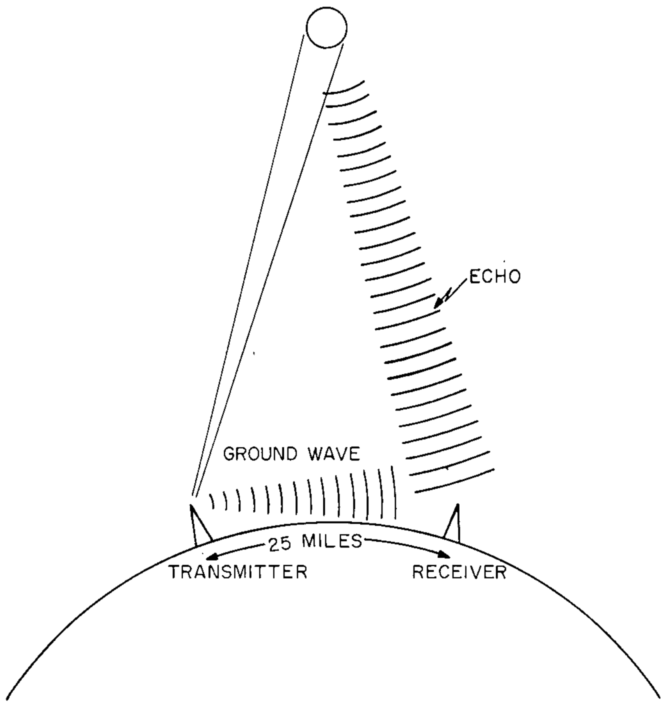

Fig. 12 - Representation of moon Doppler circuit

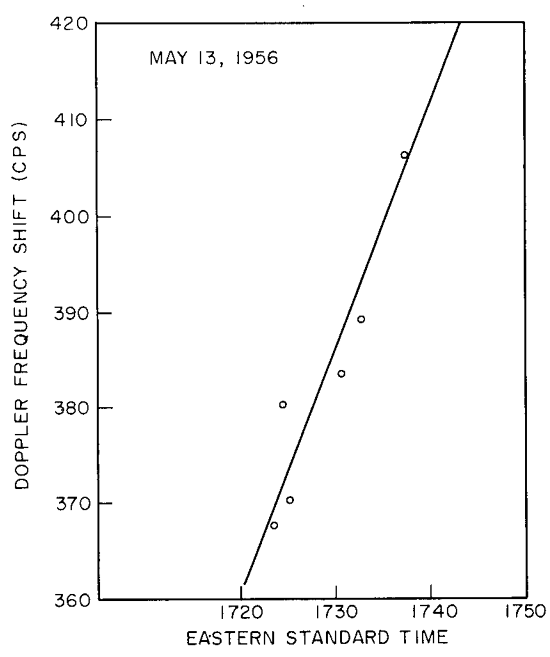

Fig. 13 - Experimentally determined Doppler frequency shift as a function of time

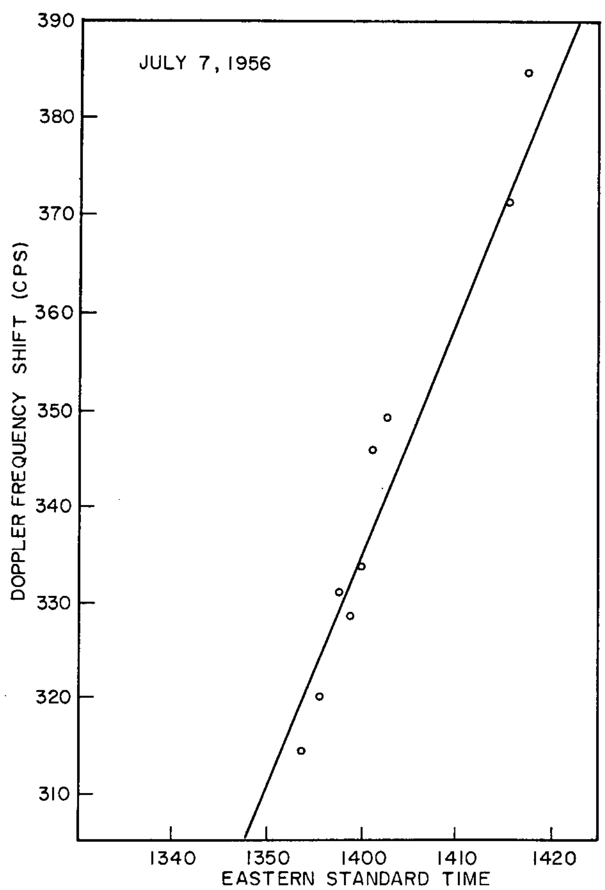

Fig. 14 - Experimentally determined Doppler frequency shift as a function of time

undergone by a radio-frequency signal after it has been reflected from the moon and received on earth. The following conclusions have been deduced from the study presented in this report:

1. There is essentially a north-south hemispheric ambiguity in Doppler position determination.

2. Latitude and longitude position errors inherent in a moon Doppler navigation system are functions of position as well as time.

3. Position errors increase with decreasing latitude (latitude position errors range from a few meters at 90-degree latitude to 100 kilometers at 0-degree latitude; and longitude position errors range from a few meters at 90-degree latitude to about 2 kilometers at 0-degree latitude), however, navigation to within line of sight of a given position (10 kilometers) may be accomplished over 70 percent of the hemisphere illuminated by the moon at a given time.

4. Position errors are independent of the transmitter frequency employed.

5. The probable error in transmitter frequency must be within the order of 1 part in 10^{11} if the full capabilities of the system are to be realized.

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This study is based on an idealized earth-moon system where the relative motion between earth site and the moon is that due to rotation of the earth alone. However, an indication that the position errors described are of the right order of magnitude is the fact that the application of the method of least squares to the limited experimental Doppler measurements yields results which agree with the calculated values of Doppler frequency shift within the computed value of the probable error.

It is apparent that the errors in position inherent in the Doppler method of position determination are large at low latitudes, however, the errors diminish at high latitudes, where some navigational systems are relatively inadequate. Furthermore, an angle-measuring method (radio sextant) might be employed to complement the Doppler method at low latitudes.

ACKNOWLEDGMENTS

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APPENDIX A
List of Magnitudes and Probable Errors of Parameters

c	$= 2.997893 \times 10^{8}$ meters/second*	(velocity of propagation)
ϵ_c	$= 5 \times 10^{2}$ meters/second	(probable error in c)
R	$= 6.378260 \times 10^{6}$ meters†	(equatorial radius of earth)
ϵ_R	$= 1 \times 10^{2}$ meters	(probable error in R)
ρ	$= 1.748 \times 10^{-2}$ radians	(horizontal parallax angle of moon, May 13)
ρ	$= 2.042 \times 10^{-2}$ radians	(horizontal parallax, July 7)
ϵ_ρ	$= 5 \times 10^{-9}$ radians	(probable error in ρ)
$d\rho/dt$	$= -5 \times 10^{-10}$ radians/second	(May 13, 1956)
	$= 8.1 \times 10^{-10}$ radians/second	(July 7, 1956)
$d^2\rho/dt^2$	$= -7.1 \times 10^{-15}$ radians/second^2	(May 13, 1956)
	$= 1.3 \times 10^{-14}$ radians/second^2	(July 7, 1956)
$\sin \rho$	$= 0.017813$	
$\cos \rho$	$= 0.99984$	
ϵ_t	$= 1 \times 10^{-3}$ seconds	(probable error in time)
δ	$= 20° \, 23'$	(declination of moon, May 13, 1956 at 2230 UT)
δ	$= 20° \, 19'$	(declination of moon, July 7, 1956 at 1900 UT)
ϵ_δ	$= 4 \times 10^{-7}$ radians	(probable error in δ)
$\cos \delta$	$= 0.93738$	(May 13, 1956)
$\sin \delta$	$= 0.34830$	
$\cos \delta$	$= 0.93779$	(July 7, 1956)
$\sin \delta$	$= 0.34721$	

*Bol and Hansen, "Microwave Cavity Determination," Stanford University, 1950
†Army Map Service, 1954

[redacted]

$L = 38.545$ degrees	(latitude of transmitter)
$\epsilon_L = 5 \times 10^{-7}$ radians	(probable error in site latitude, if surveyed by geodetic and gravimetric means)
$\cos L = 0.78212$	
$\sin L = 0.62303$	
$\omega \doteq 7 \times 10^{-5}$ radians/second	(angular velocity of earth with respect to moon)
$\epsilon_\omega = 1 \times 10^{-8}$	(probable error in ω)
$LHA = 0.557$ radians	(local hour angle of moon)
$\epsilon_{LHA} = 1 \times 10^{-6}$ radians	(probable error in LHA)
$\sin LHA = 0.55194$	
$\cos LHA \quad 0.83389$	

* * *

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APPENDIX B
Derivation of Doppler Frequency Shift and Time Rate of Change of Doppler Shift

If a receiver is in motion toward a stationary transmitter, the received frequency will be

$$f_r = f_t \ (1 + v/c) \tag{B1}$$

and if a transmitter is in motion toward a stationary receiver, the received frequency will be

$$f_r = f_t \left(\frac{1}{1 - v/c}\right) \tag{B2}$$

where

f_r is the frequency of the received signal

f_t is the frequency of the transmitted signal

c is the velocity of propagation in the medium between transmitter and receiver

v is the relative velocity of transmitter and receiver.

If the moon is in motion relative to a site on the earth, and a signal is sent from the site to the moon and there reflected, the frequency of the signal received at the earth site will be

$$f_r = f_t \ (1 + v/c) \left(\frac{1}{1 - v/c}\right). \tag{B3}$$

If the quantity, $(1 - v/c)^{-1}$, is expressed as a series and multiplied by $(1 + v/c)$, the result is

$$f_r = f_t \ (1 + 2v/c + 2v^2/c^2 + 2v^3/c^3 + \ldots) \tag{B4}$$

which, if terms of higher order than the first power are neglected, gives

$$f_r = f_t \ (1 + 2v/c), \qquad v/c < 1. \tag{B5}$$

The Doppler frequency shift is defined as the difference between the transmitted and received frequencies.

$$\Delta f = f_r - f_t \ x \ f_t \ (2v/c). \tag{B6}$$

The time rate of change of Doppler frequency shift is simply

$$\frac{d \ \Delta f}{dt} = \frac{2 \ f_t}{c} \cdot \frac{dv}{dt}. \tag{B7}$$

* * *

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APPENDIX C
Probable Error of a Computed Result

If $y = f(x)$, an error in x will produce an error in y. If the errors in x and y are denoted by Δx and Δy, the ratio of errors is $\Delta y/\Delta x$, and as Δx approaches zero $\Delta y/\Delta x$ approaches dy/dx and

$$dy = \left[\frac{dy}{dx}\right] dx. \tag{C1}$$

The differentials dy and dx represent errors in y and x respectively. When $y = f(x,z,v,\ldots)$,

$$dy = \left[\frac{\partial y}{\partial x}\right] dx + \left[\frac{\partial y}{\partial z}\right] dz + \left[\frac{\partial y}{\partial v}\right] dv + \ldots \tag{C2}$$

Squaring both sides of Eq. (C2) gives

$$\begin{aligned}(dy)^2 = {} & \left[\frac{\partial y}{\partial x}\right]^2 (dx)^2 + \left[\frac{\partial y}{\partial x} \cdot \frac{\partial y}{\partial z}\right] . dx\ dz + \left[\frac{\partial y}{\partial x} \cdot \frac{\partial y}{\partial v}\right] dx\ dv \\ & + \left[\frac{\partial y}{\partial z} \cdot \frac{\partial y}{\partial x}\right] dz\ dx \left[\frac{\partial y}{\partial z}\right]^2 (dz)^2 + \left[\frac{\partial y}{\partial z} \cdot \frac{\partial y}{\partial v}\right] dz\ dv \\ & + \left[\frac{\partial y}{\partial v} \cdot \frac{\partial y}{\partial x}\right] dv\ dx \left[\frac{\partial y}{\partial v} \cdot \frac{\partial y}{\partial z}\right] dv\ dz \left[\frac{\partial y}{\partial v}\right]^2 (dv)^2 \ldots\end{aligned} \tag{C3}$$

If the errors are symmetrically distributed about the mean, the cross products will cancel and only the squared terms will remain provided the number of sets is sufficiently large. Therefore,

$$\begin{aligned}(dy_1)^2 &= \left[\frac{\partial y}{\partial x}\right]^2 (dx_1)^2 + \left[\frac{\partial y}{\partial z}\right]^2 (dz_1)^2 + \left[\frac{\partial y}{\partial v}\right]^2 (dv_1)^2 + \ldots \\ (dy_2)^2 &= \left[\frac{\partial y}{\partial x}\right]^2 (dx_2)^2 + \left[\frac{\partial y}{\partial z}\right]^2 (dz_2)^2 + \left[\frac{\partial y}{\partial v}\right]^2 (dv_2)^2 + \ldots \\ &\ \ \vdots \\ (dy_n)^2 &= \left[\frac{\partial y}{\partial x}\right]^2 (dx_n)^2 + \left[\frac{\partial y}{\partial z}\right]^2 (dz_n)^2 + \left[\frac{\partial y}{\partial v}\right]^2 (dv_n)^2 + \ldots\end{aligned}$$

$$\sum (dy)^2 = \left[\frac{\partial y}{\partial x}\right]^2 \sum (dx)^2 + \left[\frac{\partial y}{\partial z}\right]^2 \sum (dz)^2 + \left[\frac{\partial y}{\partial v}\right]^2 \sum (dv)^2 + \ldots \tag{C4}$$

Multiplying through by $(0.6745)^2/n$ gives

$$\epsilon_y^2 = \left[\frac{\partial y}{\partial x}\right]^2 \epsilon_x^2 + \left[\frac{\partial y}{\partial z}\right]^2 \epsilon_z^2 + \left[\frac{\partial y}{\partial v}\right]^2 \epsilon_v^2 + \ldots, \tag{C5}$$

where ϵ_y is the probable error in y, and ϵ_x, ϵ_z, and ϵ_v are the probable errors in x, z, and v respectively.

The probable error of a computed result, derived from the evaluation of the probability integral

$$\phi(t) = \frac{1}{\sqrt{\pi}} \int_{-t}^{t} e^{-t^2}\, dt$$

when $\phi(t) = 0.5$, is

$$\epsilon = 0.6745\,\sigma$$

where σ is the standard deviation and is given by

$$\sigma = \sqrt{\frac{\sum d_\omega^2}{n}}.$$

Equation (C4) represents $n\sigma^2$ and when multiplied through by $(0.6745)^2/n$, an expression for the square of the probable error is obtained.

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APPENDIX D

Derivation of Relative Velocity Between Earth Site and Moon Center

A view of the earth-moon system is shown in Fig. D1

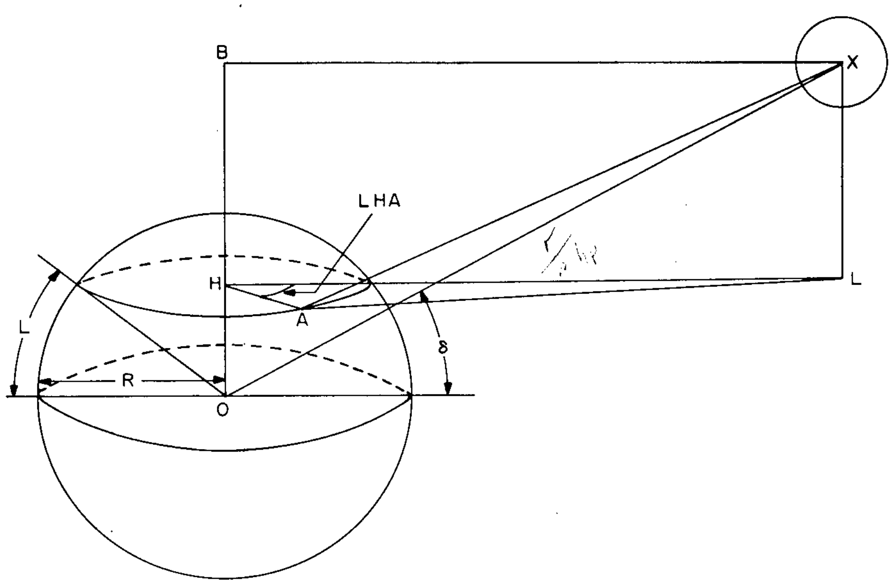

Fig. D1 - The earth-moon system

where

- A is a terrestrial site at latitude L
- O is the earth center
- X is the moon center
- δ is the declination of the moon
- R is the radius of the earth
- HA is the distance from the site to the earth's polar axis and equals $R \cos L$
- OH is the distance from the earth's equatorial plane to the plane of latitude L and equals $R \sin L$
- LHA is the local hour angle of the moon measured from site A
- OX is the distance from earth center to moon center
- OB is the distance from the earth's equatorial plane to the plane in which the moon appears to move due to the rotation of the earth and equals $OX \sin \delta$
- AX is the distance from the site to moon center
- HL is the distance from the earth's polar axis to the moon center and equals $OX \cos \delta$.

[redacted]

The equations relating the distance AX to the other quantities are constructed as follows:

$$(AX)^2 = (AL)^2 + (HB)^2 \tag{D1}$$

$$(AL)^2 = (HA)^2 + (HL)^2 - 2\,(HA)\,(HL) \cos LHA$$

$$= R^2 \cos^2 L + (OX)^2 \cos^2 \delta - 2R\,(OX) \cos L \cos \delta \cos LHA \tag{D2}$$

$$HB = OB - OH = OX \sin \delta - R \sin L. \tag{D3}$$

The substitution of Eqs. (D2) and (D3) into (D1) gives

$$(AX)^2 = R^2 \cos^2 L + (OX)^2 \cos^2 \delta - 2R\,(OX) \cos L \cos \delta \cos LHA$$
$$+ (OX)^2 \sin^2 \delta + R^2 \sin^2 L - 2R\,(OX) \sin \delta \sin L.$$

This simplifies to

$$(AX)^2 = R^2 + (OX)^2 - 2R\,(OX)\,(\cos L \cos \delta \cos LHA + \sin L \sin \delta). \tag{D4}$$

The distance AX is a function of OX and angles, LHA and δ, hence, the relative velocity between A and X is

$$v = \frac{d\,(AX)}{dt} = \left[\frac{\partial\,(AX)}{\partial\,LHA}\right]\left[\frac{d\,LHA}{dt}\right] + \left[\frac{\partial\,(AX)}{\partial\,(OX)}\right]\left[\frac{d\,(OX)}{dt}\right] + \left[\frac{\partial\,(AX)}{\partial \delta}\right]\left[\frac{d\delta}{dt}\right]. \tag{D5}$$

The individual terms of Eq. (D5) are the following:

$$\frac{\partial\,(AX)}{\partial\,LHA} = \frac{R\,(OX)\,(\cos L \cos \delta \sin LHA)}{\sqrt{R^2 + (OX)^2 - 2R\,(OX)\,(\cos L \cos \delta \cos LHA + \sin L \sin \delta)}} \tag{D6}$$

$$\frac{d\,LHA}{dt} = \omega. \tag{D7}$$

The angular velocity ω is equal to the difference between the earth's angular velocity (sidereal) and the angular velocity of the moon in its orbit, or in other words, ω is the earth's angular velocity with respect to a line connecting earth and moon centers.

$$\frac{\partial\,(AX)}{\partial\,(OX)} = \frac{OX - R(\cos L \cos \delta \cos LHA + \sin L \sin \delta)}{\sqrt{R^2 + (OX)^2 - 2R\,(OX)\,(\cos L \cos \delta \cos LHA + \sin L \sin \delta)}} \tag{D8}$$

$$\frac{d\,(OX)}{dt} = -R \cot \rho \csc \rho \frac{d\rho}{dt}. \tag{D9}$$

The determination of $d(OX)/dt$ is based on the relation

$$OX = \frac{R}{\sin \rho} \tag{D10}$$

[redacted]

where ρ is the horizontal parallax angle of the moon. This angle may be defined as the angle subtended by the earth's radius observed from the center of the moon.

$$\frac{\partial\,(AX)}{\partial\,\delta} = \frac{-R\,(\sin L \cos\delta - \cos L \cos LHA \sin\delta)\,(OX)}{\sqrt{\sin^2\rho + 1 - 2\sin\rho\,(\cos L\cos\delta\cos LHA + \sin L\sin\delta)}}\,. \tag{D11}$$

The substitution of Eqs. (D6), (D7), (D8), (D9), and (D11) into (D5) gives the following expression for relative velocity between the site A and the moon's center X

$$\begin{aligned} v = {} & \frac{\omega R\cos L\cos\delta\sin LHA}{D} \\ & + \frac{(\cos L\cos\delta\cos LHA + \sin L\sin\delta)\,R\cot\rho\,\frac{d\rho}{dt}}{D} \\ & - \frac{R\cot\rho\csc\rho\,\frac{d\rho}{dt}}{D} \\ & - \frac{(\sin L\cos\delta - \cos L\cos LHA\sin\delta)\,R\,\frac{d\delta}{dt}}{D} \end{aligned} \tag{D12}$$

where D is $\sqrt{\sin^2\rho - 2\rho\,(\cos L\cos\delta\cos LHA + \sin L\sin\delta) + 1}$. The quantity $\sin\rho$ may be replaced by ρ, since ρ is of the order of 0.02 radian, and the absolute value of the error in making the substitution is less than $\rho^3/3!$, which is less than 6 parts in a million. Similarly, $\cot\rho$ may be replaced by $1/\rho$ with an error of the same order of magnitude. Relative velocity is then expressed by

$$\begin{aligned} v = {} & \frac{R}{\sqrt{\rho^2 - 2\rho\,(\cos L\cos\delta\cos LHA + \sin L\sin\delta) + 1}}\Bigg[\omega\cos L\cos\delta\sin LHA \\ & + \left(\cos L\cos\delta\cos LHA + \sin L\sin\delta - \frac{1}{\rho}\right)\frac{1}{\rho}\,\frac{d\rho}{dt} \\ & - (\sin L\cos\delta - \cos L\cos LHA\sin\delta)\,\frac{d\delta}{dt}\Bigg]. \end{aligned} \tag{D13}$$

The term involving the time rate of change of declination is significant only when declination approaches zero, therefore, the relative velocity equation may be simplified to be

$$\begin{aligned} v \doteq {} & \frac{R}{\sqrt{\rho^2 - 2\rho\,(\cos L\cos\delta\cos LHA + \sin L\sin\delta) + 1}}\Bigg[\omega\cos L\cos\delta\sin LHA \\ & + \left(\cos L\cos\delta\cos LHA + \sin L\sin\delta - \frac{1}{\rho}\right)\frac{1}{\rho}\,\frac{d\rho}{dt} \\ & - \sin L\,\frac{d\delta}{dt}\Bigg]. \end{aligned} \tag{D14}$$

The first (rotational) term represents a sinusoidal variation having a peak value of approximately 448 meters per second and a period of 24 hours, at a latitude of zero degree. Superimposed on this variation is that due to the change in orbital distance (second term), which has a peak value of about 51 meters per second and a period of about 28 days. The third term (due to changes in declination) has a peak value of 6 meters per second and a period of about 28 days. The maximum relative contributions of the orbital and declination terms are about 11 percent and 1 percent, respectively, of that due to rotation. It is not valid to compute probable errors in relative velocity in the accustomed manner from this equation, since an error in velocity is not only a function of position but of time. Although the relative contribution of declination changes to velocity is but 1 percent of that due to rotation, there are times when the declination change is the sole source of relative velocity between earth site and moon. The same situation applies to the contribution of orbital distance changes to relative velocity.

Another point of interest is the denominator of the coefficient, or multiplying factor, of the equation. The extreme values of the denominator are 1.0002, and 0.98. If the denominator is assumed to be unity, as is sometimes done, an error of 2 percent may be realized in the computed value of relative velocity.

The derivation of an equation representing the acceleration of an earth site relative to the moon from Eq. (D14) is a formidable task, however, for the purpose of examining an extremely idealized case, the velocity equation will be simplified to be $v = \omega R \cos L \cos \delta \sin \text{LHA}$. The equation for acceleration is then simply

$$\frac{dv}{dt} = \dot{v} = \omega^2 R \cos L \cos \delta \cos \text{LHA}. \tag{D15}$$

The value of acceleration given by this equation is due to the rotation of the earth alone, but values of probable errors derived from this and the simplified velocity equation will be enlightening and give an idea of the order of magnitude of errors in position to be expected in a navigation system based on the measurement of relative velocity and acceleration. This is done in Appendixes E and F.

* * *

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APPENDIX E

Derivation of Probable Error in Computed Values of Doppler Frequency Shift and Time Rate of Change of Doppler Shift

The Doppler frequency shift (Appendix B) is given by $\Delta f = 2 f_t v/c$, therefore, the probable error in Δf (Appendix C) is

$$\epsilon_{\Delta f} = \sqrt{\left[\frac{\partial \Delta f}{\partial f_t}\right]^2 \epsilon_{ft}^{2} + \left[\frac{\partial \Delta f}{\partial v}\right]^2 \epsilon_v^{2} + \left[\frac{\partial \Delta f}{\partial c}\right]^2 \epsilon_c^{2}}\,. \tag{E1}$$

Performing the indicated differentiation and simplifying gives

$$\epsilon_{\Delta f} = \Delta f \sqrt{\left(\frac{\epsilon_{ft}}{f_t}\right)^2 + \left(\frac{\epsilon_v}{v}\right)^2 + \left(\frac{\epsilon_c}{c}\right)^2}\,. \tag{E2}$$

The time rate of change of Doppler shift is given by $\dot{\Delta f} = 2 f_t/c \quad \dot{v}$, and an analogous operation gives

$$\epsilon_{\dot{\Delta f}} = \dot{\Delta f} \sqrt{\left(\frac{\epsilon_{ft}}{f_t}\right)^2 + \left(\frac{\epsilon_{\dot{v}}}{\dot{v}}\right)^2 + \left(\frac{\epsilon_c}{c}\right)^2}\,. \tag{E3}$$

It is of interest to note that the quantities actually measured by an observer at the earth site are not Δf and $\dot{\Delta f}$. The quantities measured are, if Δf is determined by beating the transmitter frequency with the echo,

$$\Delta f' = f_r - f_t(t+\Delta t)$$

$$\dot{\Delta f}' = \frac{d\,\Delta f'}{dt}.$$

The quantity f_r (Appendix B) is

$$f_r = f_t + \frac{2 f_t v}{c}$$

hence

$$\Delta f' = \Delta f - \Delta f_t \tag{E4}$$

where Δf_t is the change in transmitter frequency during the transit time of the signal from earth to moon to earth.

The relative velocity of a site on the earth with respect to the moon due to rotation of the earth (Appendix D) has been shown to be $v = \omega R \cos L \cos \delta \sin LHA$. The probable error in relative velocity is

$$\epsilon_v = \left(\left[\frac{\partial v}{\partial R}\right]^2 \epsilon_R^{2} + \left[\frac{\partial v}{\partial t}\right]^2 \epsilon_t^{2} + \left[\frac{\partial v}{\partial \omega}\right]^2 \epsilon_\omega^{2} + \left|\frac{\partial v}{\partial \delta}\right|^2 \epsilon_\delta^{2} + \left|\frac{\partial v}{\partial L}\right|^2 \epsilon_L^{2}\right)^{1/2}. \tag{E5}$$

[redacted]

The partial derivatives are:

$$\frac{\partial v}{\partial R} = \omega \cos L \cos \delta \sin LHA \tag{E6}$$

$$\frac{\partial v}{\partial t} = \omega^2 R \cos L \cos \delta \cos LHA \tag{E7}$$

$$\frac{\partial v}{\partial \omega} = R \cos L \cos \delta \, (\sin LHA + LHA \cos LHA) \tag{E8}$$

$$\frac{\partial v}{\partial \delta} = -\omega R \sin \delta \cos L \sin LHA \tag{E9}$$

$$\frac{\partial v}{\partial L} = -\omega R \cos \delta \sin L \sin LHA. \tag{E10}$$

The relative acceleration of a site on earth with respect to the moon due to rotation of the earth (Appendix D) is $\dot{v} = \omega^2 R \cos L \cos \delta \cos LHA$. The probable error in acceleration is

$$\epsilon_{\dot{v}} = \left(\left[\frac{\partial \dot{v}}{\partial R}\right]^2 \epsilon_R{}^2 + \left[\frac{\partial \dot{v}}{\partial t}\right]^2 \epsilon_t{}^2 + \left[\frac{\partial \dot{v}}{\partial \omega}\right]^2 \epsilon_\omega{}^2 + \left[\frac{\partial \dot{v}}{\partial \delta}\right]^2 \epsilon_\delta{}^2 + \left[\frac{\partial \dot{v}}{\partial L}\right]^2 \epsilon_L{}^2 \right)^{1/2}. \tag{E11}$$

The partial derivatives are:

$$\frac{\partial \dot{v}}{\partial R} = \omega^2 \cos L \cos \delta \cos LHA \tag{E12}$$

$$\frac{\partial \dot{v}}{\partial t} = -\omega^3 R \cos L \cos \delta \sin LHA \tag{E13}$$

$$\frac{\partial \dot{v}}{\partial \omega} = R \cos L \cos \delta \, (2\omega \cos LHA - \omega^3 \sin LHA) \tag{E14}$$

$$\frac{\partial \dot{v}}{\partial \delta} = -\omega^2 R \cos L \sin \delta \cos LHA \tag{E15}$$

$$\frac{\partial \dot{v}}{\partial L} = -\omega^2 R \sin L \cos \delta \cos LHA. \tag{E16}$$

[redacted]

It is more meaningful to show the product of each of the partial derivatives with the appropriate probable error.

$$\frac{\partial v}{\partial R} \epsilon_R \approx 7 \times 10^{-3} \cos L \cos \delta \sin \mathrm{LHA}$$

$$\frac{\partial v}{\partial t} \epsilon_t \approx 3.1 \times 10^{-5} \cos L \cos \delta \cos \mathrm{LHA}$$

$$\frac{\partial v}{\partial \omega} \epsilon_\omega \approx 6.4 \times 10^{-2} \cos L \cos \delta (\sin \mathrm{LHA} + \mathrm{LHA} \cos \mathrm{LHA})$$

$$\frac{\partial v}{\partial \delta} \epsilon_\delta \approx 1.8 \times 10^{-4} \sin \delta \cos L \sin \mathrm{LHA}$$

$$\frac{\partial v}{\partial L} \epsilon_L \approx 2.2 \times 10^{-4} \cos \delta \sin L \sin \mathrm{LHA}$$

$$\frac{\partial \dot{v}}{\partial R} \epsilon_R \approx 4.9 \times 10^{-7} \cos L \cos \delta \cos \mathrm{LHA}$$

$$\frac{\partial \dot{v}}{\partial t} \epsilon_t \approx 2.2 \times 10^{-9} \cos L \cos \delta \sin \mathrm{LHA}$$

$$\frac{\partial \dot{v}}{\partial \omega} \epsilon_\omega \approx 4.5 \times 10^{-6} \cos L \cos \delta (2 \cos \mathrm{LHA} - 4.9 \times 10^{-9} \sin \mathrm{LHA})$$

$$\frac{\partial \dot{v}}{\partial \delta} \epsilon_\delta \approx 1.2 \times 10^{-8} \cos L \sin \delta \cos \mathrm{LHA}$$

$$\frac{\partial \dot{v}}{\partial L} \epsilon_L \approx 1.6 \times 10^{-8} \sin L \cos \delta \cos \mathrm{LHA}.$$

* * *

[redacted]

[redacted]

APPENDIX F
Derivation of Probable Errors in Position

The general equation of a sphere of radius r in a rectangular coordinate system is $x^2 + y^2 + z^2 = r^2$

where

$$x = r \cos \phi \cos \theta$$

$$y = r \cos \phi \sin \theta$$

$$z = r \sin \phi$$

$$\theta = \arctan \frac{y}{x}$$

$$\phi = \arccos \sqrt{\frac{x^2 + y^2}{r}} \; .$$

The sphere described by these equations is shown in Fig. F1.

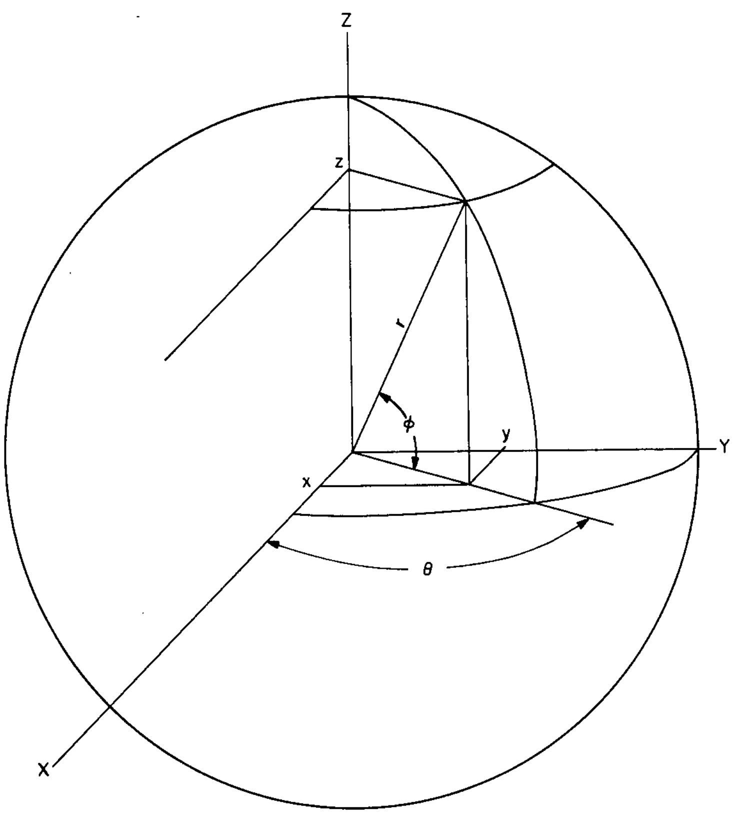

Fig. F1

[redacted]

The error in position on the surface of the sphere is $r\Delta\phi$ in the ϕ direction, and $r \cos \phi \, \Delta\theta$ in the θ direction. The angles ϕ and θ are functions of x and y, therefore, the incremental angles $\Delta\phi$ and $\Delta\theta$ are found in the following manner:

$$\cos \phi = \sqrt{\frac{x^2 + y^2}{r^2}}$$

$$\cos (\phi + \Delta\phi) = \sqrt{\frac{(x + \Delta x)^2 + (y + \Delta y)^2}{r^2}}$$

$$\cos \phi \cos \Delta\phi - \sin \phi \sin \Delta\phi = \sqrt{\frac{x^2 + y^2}{r^2} + \frac{2\,(x\,\Delta x + y\,\Delta y)}{r^2} + \frac{\Delta x^2 + \Delta y^2}{r^2}}. \qquad \text{(F1)}$$

If higher orders than the first of Δx and Δy are neglected and substitutions for x and y are made, the result is

$$\cos \phi \cos \Delta\phi - \sin \phi \sin \Delta\phi = \sqrt{\cos^2 \phi + \frac{2 \cos \phi}{r} (\cos \theta \Delta x + \sin \theta \Delta y)}$$

or

$$\sin \phi \sin \Delta\phi = [\cos \phi] \left[\cos \Delta\phi - \sqrt{1 + \frac{2\,(\cos \theta \Delta x + \sin \theta\, \Delta y)}{r \cos \phi}} \right]. \qquad \text{(F2)}$$

When $\Delta\phi$ approaches zero, Eq. (F1) becomes

$$\sin \phi \, d\phi = [\cos \phi] \left[1 - \sqrt{1 + \frac{2\,(\cos \theta\, dx + \sin \theta\, dy)}{r \cos \phi}} \right],$$

and since $\sqrt{1 + A} \approx 1 + A/2$ when A is very small,

$$\sin \phi \, d\phi = [\cos \phi] \left[\frac{-\,(\cos \theta\, dx + \sin \theta\, dy)}{r \cos \phi} \right]$$

and

$$d\phi = \frac{-\,(\cos \theta\, dx + \sin \theta\, dy)}{r \sin \phi}. \qquad \text{(F3)}$$

This is the angular error in the ϕ direction as a function of x, y, and θ, and may be used when $\phi > \Delta\phi$. However, when ϕ becomes small or approaches zero, Eq. (F1) becomes

$$\cos \Delta\phi = \sqrt{1 + \frac{2\,(\cos \theta\, \Delta x + \sin \theta\, \Delta y)}{r}}$$

$$\cos^2 \Delta\phi = 1 + \frac{2}{r} (\cos \theta\, \Delta x + \sin \theta\, \Delta y)$$

$$\sin^2 \Delta\phi = \frac{-2}{r} (\cos\theta\,\Delta x + \sin\theta\,\Delta y)$$

or

$$\sin \Delta\phi \doteq \Delta\phi = \sqrt{\frac{-2}{r} (\cos\theta\,\Delta x + \sin\theta\,\Delta y)}\,. \tag{F4}$$

This is the expression for the determination of the angular error in ϕ for values of ϕ near zero.

The angular error in the θ direction, $\Delta\theta$, is determined as follows:

$$\tan\theta = \frac{y}{x}$$

$$\tan(\theta + \Delta\theta) = \frac{y + \Delta y}{x + \Delta x}$$

$$\frac{\tan\theta + \tan\Delta\theta}{1 - \tan\theta\tan\Delta\theta} = \frac{y + \Delta y}{x + \Delta x}$$

and replacing $\tan\Delta\theta$ by $\Delta\theta$,

$$(\Delta\theta + \tan\theta)(x + \Delta x) = (1 - \Delta\theta\tan\theta)(y + \Delta y)$$

$$\Delta\theta = \frac{y + \Delta y - (x + \Delta x)\tan\theta}{x + \Delta x + (y + \Delta y)\tan\theta}$$

$$\Delta\theta = \frac{r\cos\phi\sin\theta + \Delta y - \dfrac{\sin\theta}{\cos\theta}(r\cos\phi\cos\theta + \Delta x)}{r\cos\phi\cos\theta + \Delta x + \dfrac{\sin\theta}{\cos\theta}(r\cos\phi\sin\theta + \Delta y)}$$

which reduces to

$$\Delta\theta = \frac{\cos\theta\,\Delta y - \sin\theta\,\Delta x}{r\cos\phi + \cos\theta\,\Delta x + \sin\theta\,\Delta y}\,. \tag{F5}$$

In the general case, where ϕ does not approach 90 degrees, $\Delta\theta$ may be represented by

$$\Delta\theta = \frac{\cos\theta\,\Delta y - \sin\theta\,\Delta x}{r\cos\phi}\,. \tag{F6}$$

The limiting value of $\Delta\theta$ when ϕ approaches 90 degrees is

$$\lim_{\phi\to 90^\circ} \Delta\theta = \frac{\cos\theta\,\Delta y - \sin\theta\,\Delta x}{\cos\theta\,\Delta x + \sin\theta\,\Delta y}\,. \tag{F7}$$

The equations describing the relative velocity and acceleration of an earth site with respect to the moon due only to the rotation of the earth (Appendix D) are

$$v = \omega R\cos\delta\cos L\sin LHA$$

and

$$\dot{v} = \omega^2 R\cos\delta\cos L\cos LHA\,.$$

The Doppler frequency shift and the time rate of change of Doppler shift (Appendix B) are given by

$$\Delta f = \frac{2\ f_t\ v}{c} = \frac{2\ f_t\ \omega R\ \cos\ \delta\ \cos\ L\ \sin\ LHA}{c}$$

and

$$\dot{\Delta f} = \frac{2\ f_t\ \dot{v}}{c} = \frac{2\ f_t\ \omega^2 R\ \cos\ \delta\ \cos\ L\ \cos\ LHA}{c}.$$

When Δf is multiplied by ω, the equations for $\omega\ \Delta f$ and $\dot{\Delta f}$ describe a sphere

where

$$x = \dot{\Delta f}$$

$$y = \omega\ \Delta f$$

$\Delta x = \epsilon_{\dot{\Delta f}}$ (probable error in the time rate of change of Doppler shift)

$\Delta y = \epsilon_{\omega \Delta f} = \sqrt{\omega^2\ \epsilon_{\Delta f}^{\ 2} + \Delta f^2\ \epsilon_{\omega}^{\ 2}}$ (probable error in the product of ω and Δf)

$\phi = L$ (latitude)

$\theta = LHA$ (local hour angle)

$$r = \frac{2\ f_t\ \omega^2 R\ \cos\ \delta}{c}.$$

The substitution of these quantities in Eqs. (F3), (F4), (F5), and (F7) gives expressions for the angular errors in latitude and longitude.

The probable position error in latitude (in meters) is given by the product of the angular error in latitude and the radius of the earth. Using Eq. (F3)

$$R\ dL = \frac{-\ c\left(\epsilon_{\dot{\Delta f}}\ \cos\ LHA + \sin\ LHA\ \sqrt{\omega^2\ \epsilon_{\Delta f}^{\ 2} + \Delta f^2\ \epsilon_{\omega}^{\ 2}}\right)}{2\ f_t\ \omega^2\ \cos\ \delta\ \sin\ L} \tag{F8}$$

which approaches Eq. (F4) as a limit, when latitude approaches zero,

$$R\ \Delta L = \sqrt{\frac{-\ c\ R\ \ \epsilon_{\dot{\Delta f}}\ \cos\ LHA + \sin\ LHA\ \sqrt{\omega^2\ \epsilon_{\Delta f}^{\ 2} + \Delta f^2\ \epsilon_{\omega}^{\ 2}}}{f_t\ \omega^2\ \cos\ \delta}}. \tag{F9}$$

The probable position error in longitude (in meters) is given by the product of the angular error in local hour angle (ΔLHA), the earth's radius and the cosine of the latitude. Using Eq. (F6)

$$R(\cos\ L)\ \Delta LHA = \frac{c\left(\cos\ LHA\ \sqrt{\omega^2\ \epsilon_{\Delta f}^{\ 2} + \Delta f^2\ \epsilon_{\omega}^{\ 2}} - \epsilon_{\dot{\Delta f}}\ \sin\ LHA\right)}{2\ f_t\ \omega^2\ \cos\ \delta}. \tag{F10}$$